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Maximally-Mixed Three Generations of Neutrinos and the Solar and Atmospheric Neutrino Problems

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Abstract

Motivated by the indication that both the Solar and the atmospheric neutrino puzzles may simultaneously be solved by (vacuum as well as matter-induced resonant) oscillations of two generations of neutrinos with large mixing, we have analyzed the data on the Solar and atmospheric neutrinos assuming that all *three* neutrinos are maximally mixed. It is shown that the results of two-generation analyses are still valid even in the three-generation scheme, *i.e.*, the two puzzles can be solved simultaneously if $\Delta m^2_{21} = m^2_2 - m^2_1 \simeq 10^{-10}\text{eV}^2$ and $\Delta m^2_{31} = 10^{-3} \sim 10^{-1}\text{eV}^2$. We have also demonstrated explicitly that with the use of the see-saw mechanism it is possible to have large or maximal mixings for neutrinos even though their masses are highly non-degenerate.

1 Introduction

It has long been known that the measured Solar neutrino event rates by three types of experimental detectors [1, 2, 3, 4] all have shown the deficiency compared with the Standard Solar Model (SSM) predictions by Bahcall and others [5, 6]. The latest ratios of the observed event rates and the SSM predictions are [7]

$$\begin{aligned}
R_{Ga}(\text{GALLEX}) &= 0.66 \pm 0.10 \\
R_{Ga}(\text{SAGE}) &= 0.44^{+0.17}_{-0.21} \\
R_{Cl} &= 0.26 \pm 0.05 \\
R_{Kam} &= 0.47 \pm 0.09 \ .
\end{aligned} \tag{1}$$

The deviation of R from unity is called the Solar neutrino problem or puzzle.

Another important source of non-accelerator neutrinos is the cosmic rays. In addition to ν_e, ν_μ and $\overline{\nu_e}$, $\overline{\nu_\mu}$ from $\pi^\pm$ and $K^\pm$ decay, the decay-product muons can also produce muon and electron neutrinos within the atmosphere and the resulting neutrinos are detected after traveling through the atmosphere or the Earth. They are the atmospheric neutrinos. The ratio of the ν_μ (and $\overline{\nu_\mu}$) flux to the ν_e (and $\overline{\nu_e}$) flux is roughly two which has been confirmed by detailed Monte Carlo calculations [8] for low energy atmospheric neutrinos ($0.1\text{GeV} \lesssim E_\nu \lesssim 2\text{GeV}$). Two large underground water Cherenkov detectors, the IMB [9] and Kamiokande [10], found the ratios, for the contained events,

$$R\left(\frac{\mu}{e}\right) = \frac{R\left(\frac{\mu}{e}\right)_{\text{exp}}}{R\left(\frac{\mu}{e}\right)_{\text{MC}}} = \begin{cases} 0.60^{+0.07}_{-0.06} \pm 0.05 & \text{Kamiokande} \\ 0.54 \pm 0.05 \pm 0.12 & \text{IMB}\ , \end{cases} \tag{2}$$

where $R\left(\mu/e\right)_{\text{exp}}$ and $R\left(\mu/e\right)_{\text{MC}}$ are the observed and Monte Carlo simulated ratios, respectively, for the muon and electron events induced in the detectors by ν_μ ($\overline{\nu_\mu}$) and

ν_e ($\overline{\nu_e}$). The preliminary data from SOUDAN-2 [11] also supports the above results with $R(\mu/e) = 0.55 \pm 0.27 \pm 0.10$, whereas the previous Frejus [12]and NUSEX [13] experiments failed to see the deviation of $R(\mu/e)_{\rm exp}$ from unity. We wish to mention that this anomaly, in particular the Kamioka data, has about the same statistical and systematic significance as in the case of the Solar neutrino deficit, and, furthermore, is less model dependent than the Solar neutrino case. (However, it has been argued by some that unlike the Solar neutrinos, the electron and muon identification for the atmospheric neutrino experiments needs more improvement.)

If we take both anomalies seriously, one of the popular and plausible solutions is the neutrino oscillation.

In the case of the Solar neutrinos, the matter-enhanced neutrino oscillations (the Mikheyev-Smirnov-Wolfenstein (MSW) effect [14]) can explain the problem, yielding two distinct allowed regions in the Δm^2-$\sin^2 2\theta$ parameter space. One of the solutions is the so-called large $\sin^2(2\theta)$ solution with $\Delta m^2 \sim 10^{-5}\text{eV}^2$. The validity of this solution has been questioned in the past based on the poor χ^2 fit. However, this criticism is unwarranted because analyses using the three generations of neutrinos enlarge the allowed area of the large angle solution considerably [15].

The explanation [16, 17, 18] based on the long-wavelength vacuum oscillations, on the other hand, is still viable and leads to the allowed region of the $\Delta m^2 - \sin^2 2\theta$ plot to be around $\Delta m^2 \simeq 10^{-10}(\text{eV})^2$ and $0.75 \lesssim \sin^2 2\theta \lesssim 1.0$.

The most intriguing explanation of the atmospheric neutrino problem is also due to neutrino oscillations in vacuum, *i.e.*, the ν_μ ($\overline{\nu_\mu}$) is converted into ν_τ ($\overline{\nu_\tau}$) depleting its flux whereas the ν_e ($\overline{\nu_e}$) flux remains unchanged. This explanation again yields a solution with a large $\sin^2(2\theta)$ value in the $\Delta m^2 - \sin^2(2\theta)$ plot with $10^{-3}\text{eV}^2 \lesssim \Delta m^2 \lesssim 10^{-1}\text{eV}^2$.

We note that the common feature of these neutrino oscillation solutions is the large mixing between two neutrinos and the results remain valid as long as the third generation neutrino is magically decoupled while the other two neutrinos are almost maximally mixed. In general, however, we have to include all three neutrino generations with arbitrary mixing and mass parameters for fully consistent analysis.

In this paper, however, in order to make the analysis simple, we assume that the three generation neutrinos are maximally mixed and keep only the mass parameters free. The maximal mixing provides us with great mathematical simplicity. However, this does not imply the loss of generality because as long as mixing angles are reasonably large, the qualitative results still remain the same. The three generation neutrino oscillation with maximal mixing has been discussed in the past [17], but the authors did not take into account of the energy dependence of the oscillating term by restricting the mass squared differences to be greater than $10^{-10}(\mathrm{eV})^2$. In this case the survival probability for the solar neutrino, $P(\nu_e \rightarrow \nu_e)$, at all energies, is 1/3. The atmospheric neutrino anomaly has also been addressed by them by assuming that all Δm_{ij}^2 to be in the range $(0.5 - 1.2) \times 10^{-2}\mathrm{eV}^2$.

Here, we do not make any assumption on the mass parameters and keep the energy dependence of the survival probability when we calculate the event rates for the Solar neutrino experiments and flux ratios for the atmospheric neutrino experiments. We demonstrate that, as hinted in the two generation analysis, as long as $\Delta m_{21}^2 \sim 10^{-10}\mathrm{eV}^2$ and $\Delta m_{32}^2 = (10^{-3} \sim 10^{-1})\mathrm{eV}^2$, the maximally mixed three generations of neutrino scheme is consistent with all the data for the Solar and atmospheric neutrinos. Finally, we show that large neutrino mixing angles with completely non-degenerate neutrino mass hierarchy can be realized in the frame work of the see-saw mechanism.

2 Maximal Mixing among Three Neutrinos

The oscillation probability for $\nu_\alpha \to \nu_\beta$ (α, β are flavor indices) for the three generation case is given by [19]

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= (a+b-c) \mid U^*_{\alpha 1} \mid^2 \mid U_{\beta 1} \mid^2 + (a-b+c) \mid U^*_{\alpha 2} \mid^2 \mid U_{\beta 2} \mid^2 \\ &\quad + (-a+b+c) \mid U^*_{\alpha 3} \mid^2 \mid U_{\beta 3} \mid^2 , \end{aligned} \tag{3}$$

where $U_{\alpha i}$ and $U_{\beta i}$ are the matrix elements of a 3×3 unitary matrix which relates the weak and mass eigenstates of neutrinos. In Eq. (3)

$$\begin{aligned} a &\equiv 2\sin^2\left(\frac{1.27 L \Delta m^2_{21}}{E}\right) \\ b &\equiv 2\sin^2\left(\frac{1.27 L \Delta m^2_{31}}{E}\right) \\ c &\equiv 2\sin^2\left(\frac{1.27 L \Delta m^2_{32}}{E}\right) , \end{aligned} \tag{4}$$

where L, E and $\Delta m^2_{ij} \equiv m^2_i - m^2_j$ are in units of meter, MeV and eV^2, respectively.

We are interested in the case where three neutrinos are maximally mixed, *i.e.*, the mixing matrix U is given by [20, 21]:

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -x & -x^2 \\ 1 & -x^2 & -x \end{pmatrix} , \; x = e^{2\pi i/3} . \tag{5}$$

Neutrino masses are left unconstrained in the model under discussion. As can be seen in Eq. (5), CP must be violated; otherwise U is not unitary. By substituting Eq. (5) into Eq. (3), we find

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= P(\nu_\mu \to \nu_\tau) = P(\nu_e \to \nu_\tau) \\ &= \frac{2}{9}\left(\sin^2 k_{21} + \sin^2 k_{31} + \sin^2 k_{32}\right) , \end{aligned} \tag{6}$$

where

$$k_{ij} \equiv \frac{1.27 L \Delta m_{ij}^2}{E} \ . \tag{7}$$

As implied by maximal mixing, the three generations of neutrinos oscillate into each other with equal probability. It should be noted that as long as neutrino masses are sufficiently small, the maximal mixing is phenomenologically consistent with all the previously known constraints.

2.1 Solar Neutrinos

For the solar neutrinos the distance involved is given by

$$L = L_0 \left[1 - \epsilon \cos \left(2\pi \frac{t}{T} \right) \right] , \tag{8}$$

where L_0 is one Astronomical Unit ($= 1.5 \times 10^{11}$m), $\epsilon = 0.0167$ is the eccentricity of the Earth orbit and t takes $t = 0$ on June 21 ($T = 365$ days). Since we are anticipating (see below) $\Delta m_{31}^2 \simeq \Delta m_{32}^2 \gg 10^{-10}\mathrm{eV}^2$ in order to solve the atmospheric neutrino problem, we have, from Eq. (6),

$$P(\nu_e \to \nu_\mu) \ = \ P(\nu_e \to \nu_\tau) \ = \ \frac{2}{9} \left(\sin^2 k_{21} + \frac{1}{2} + \frac{1}{2} \right) , \tag{9}$$

or

$$P(\nu_e \to \nu_e) \ = \ 1 - \frac{4}{9} \left(1 + \sin^2 k_{21} \right) \ = \ \frac{5}{9} - \frac{4}{9} \sin^2 k_{21} \ . \tag{10}$$

Note that the maximal value of the survival probability of ν_e is $(5/9) = 0.56$. This implies that, since the Standard Solar Model predicts the event rate for the GALLEX detector to be 132^{+9}_{-6} SNU, the observed event rate must be

$$\Sigma_{\mathrm{expt}} \lesssim 80 \ . \tag{11}$$

If the $\Sigma_{\rm expt}$ turns out to be greater than the above, this maximal mixing model (as an explanation with vacuum oscillation) is ruled out (or Δm^2_{32} and Δm^2_{31} must take values such that $\sin^2 k_{32}$ and $\sin^2 k_{31}$ cannot be approximated by 1/2). The latest GALLEX rate is $\Sigma_{\rm expt} = 87 \pm 14 \pm 7$ which is still consistent with Eq. (11). By using the formula for the observed event rate

$$\Sigma = \sum_i \int \sigma_i(E_\nu)\phi_i(E_\nu)P(\nu_e \rightarrow \nu_e; E_\nu)dE_\nu, \tag{12}$$

where the summation is for all possible neutrino-production reactions such as pp, ^{7}Be, ^{8}B, $\cdots$ processes, $\sigma_i(E)$ are the detection cross sections and ϕ_i are the SSM neutrino fluxes, we have carried out numerical calculation of Eq. (12). The results of the calculations for the Kamioka, Cl and Ga experiments are shown in Fig. (2.1). In the figures, the horizontal solid lines are the central values of the experiments with the dashed lines representing one standard deviation. Oscillating solid lines represent the theoretical values obtained from Eq. (12), again with the dotted lines indicating one standard deviation resulting from the errors in the SSM. We can see that in the region of Δm^2_{21}

$$4 \times 10^{-11}{\rm eV}^2 \lesssim \Delta m^2_{21} \lesssim 2 \times 10^{-10}{\rm eV}^2 \ , \tag{13}$$

theory and experiments agree within one standard deviation. This conclusion is in agreement with that of the two generation analysis.

2.2 Atmospheric Neutrinos

Let N_e and N_μ be the original ν_e and ν_μ fluxes, respectively, at the point of production somewhere in the atmosphere. Now, due to oscillations after travelling a distance L, the effective flux at the point of detection is

$$\begin{aligned}
N_e^{eff} &= N_e + N_\mu P(\nu_\mu \to \nu_e) - N_e P(\nu_e \to \nu_\mu) - N_e P(\nu_e \to \nu_\tau) \\
&= N_e \left[1 - P(\nu_e \to \nu_\mu) - P(\nu_e \to \nu_\tau) + \left(\frac{N_\mu}{N_e}\right) P(\nu_\mu \to \nu_e)\right] ,
\end{aligned} \tag{14}$$

where $(N_\mu/N_e) \simeq 2.08$ is the calculated ratio of ν_μ and ν_e for low energy neutrinos [8].

Since we have $L \lesssim 13 \times 10^3$ Km, $E_\nu \gtrsim 0.2$ GeV and $\Delta m^2_{21} \sim 10^{-10}$ eV^2, $\sin^2 k_{21}$ is well approximated by zero and, because of the assumed hierarchy $m_3 > m_2 > m_1$, we expect

$$\sin^2 k_{32} \simeq \sin^2 k_{31} , \tag{15}$$

so that

$$\begin{aligned}
P(\nu_e \to \nu_\mu) &= P(\nu_e \to \nu_\tau) = \frac{2}{9}\left(\sin^2 k_{21} + \sin^2 k_{32} + \sin^2 k_{31}\right) \\
&\simeq \frac{4}{9} \sin^2 k_{32} \equiv P .
\end{aligned} \tag{16}$$

Therefore, the ratio of the effective and original fluxes is given by

$$R(\nu_e \to \nu_e) \equiv \left(\frac{N_e^{eff}}{N_e}\right) = 1 - 2P + 2.08P = 1 + 0.08P \simeq 1 , \tag{17}$$

implying that the maximal mixing oscillations practically do not modify the ν_e flux. It is interesting that in our maximal mixing scheme, the electron neutrino flux naturally remains unchanged in spite of equal oscillation probabilities among ν_e, ν_μ and ν_τ.

In the case of ν_μ, we have, using the same notation as above,

$$\begin{aligned}
N_\mu^{eff} &= N_\mu + N_e P(\nu_e \to \nu_\mu) - N_\mu P(\nu_\mu \to \nu_e) - N_\mu P(\nu_\mu \to \nu_\tau) \\
&= N_\mu \left[1 - 2\ P + 0.48\ P\right] \ = \ N_\mu \left(1 - 1.52\ P\right),
\end{aligned} \tag{18}$$

leading to

$$R\,(\nu_\mu \to \nu_\mu) = \left(\frac{N_\mu^{eff}}{N_\mu}\right) \; = \; 1 - 1.52 \; P \; . \tag{19}$$

Since we have $P \leq 4/9$ [see Eq. (10)], the maximal mixing oscillations can deplete ν_μ by as much as 80%. From Eqs. (17) and (19) the effective ν_μ and ν_e flux ratio is given by

$$\frac{R(\nu_\mu \to \nu_\mu)}{R(\nu_e \to \nu_e)} = \frac{1 - 1.52P}{1 + 0.08P} \simeq 1 - 1.6P \; . \tag{20}$$

The observed μ/e ratio at the Kamiokande in which most of theoretical uncertainties in the Monte Carlo calculations are cancelled out is 0.60 for the "contained events". This implies, in this naive estimate, that, by setting the right-hand side of Eq. (20) equal to 0.60,

$$P(\nu_\mu \to \nu_\tau) = \frac{4}{9} \sin^2 \left(1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle \right) \simeq 0.25 \; . \tag{21}$$

or

$$\sin^2 \left(1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle \right) \simeq 0.56 \; .$$

It is interesting to note that the required value of $\sin^2 \left(1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle \right)$ to explain the data is very close to 1/2. This can be realized in two ways. First, the oscillation is already in the rapid oscillation region, leading to $\sin^2 \left(1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle \right) \simeq 1/2$ which implies $1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle$ is much larger than unity. Since $\left\langle \frac{L}{E_\nu} \right\rangle \gtrsim 10^2 (\frac{\mathrm{Km}}{\mathrm{GeV}})$, we have $\Delta m_{32}^2 \gtrsim 10^{-2} \mathrm{eV}^2$. Another possibility is that $\sin^2 \left(1.27 \Delta m_{32}^2 \left\langle \frac{L}{E_\nu} \right\rangle \right)$ happens to be just 0.56. Since the average value of $\langle L/E_\nu \rangle$ for the fully contained events is estimated to be $10^2 (\frac{\mathrm{Km}}{\mathrm{GeV}}) \lesssim \langle \frac{L}{E_\nu} \rangle \lesssim 10^3 (\frac{\mathrm{Km}}{\mathrm{GeV}})$, we find, in this case,

$$10^{-3} \mathrm{eV}^2 \lesssim \Delta m_{32}^2 \lesssim 10^{-2} \mathrm{eV}^2 \; . \tag{22}$$

Combining both possibilities (which we cannot distinguish at present) and the fact that $\Delta m_{32}^2 \gtrsim (10^{-1} \sim 1) \mathrm{eV}^2$ is already eliminated for the large mixing case from laboratory oscillation experiments, we can conclude that

$$10^{-3}\mathrm{eV}^2 \lesssim \Delta m_{32}^2 \lesssim (10^{-1} \sim 1)\mathrm{eV}^2 \; . \tag{23}$$

The calculated flux for the upward through-going muons is subject to uncertainties in the cosmic ray flux. It has been shown in Ref. [22] that the ratio of the observed number of such muons and the calculated values ranges from 0.84 to 0.94. If we interpret the deviation of these ratios from unity as an indication of the ν_μ depletion due to oscillations, we have, from Eq. (19),

$$(0.84 \sim 0.94) = 1 - 1.52 \cdot \frac{4}{9} \sin^2 \left(1.27 \Delta m_{32}^2 \times 10^2 \right) \; , \tag{24}$$

where we have used the value $\langle L/E_\nu \rangle \simeq 10^2$ (Km/GeV) [22] and Δm_{32}^2 is in units of eV^2. Equation (24) yields, with the constraint given by Eq. (23),

$$\Delta m_{32}^2 = (2.47\ n + 0.2 \sim 2.47\ n + 0.4) \times 10^{-2}\ \mathrm{eV}^2 \; , \tag{25}$$

where $n = 0, 1, 2, \cdots$.

In contrast to the case we have just discussed, the ratio of the upward stopping muons and the upward through-going muons is practically free of the flux normalization uncertainties. The calculated ratio for the Kamiokande detector ranges from 0.28 to 0.30 [22], indicating the insensitivity to the method of calculations. Unfortunately, the relevant experimental data are not yet available from Kamiokande. A preliminary analysis of the IMB-3 group has not come up with any obvious discrepancy between the observed and calculated ratios.

If our interpretation of ν_μ oscillations is valid, we expect the observed ratio to be different from the above calculated ratio by a factor

$$R\left(\frac{S}{T}\right) = \frac{1 - 1.52P\left(\left\langle \frac{L}{E_\nu} \right\rangle = 10^3 \left(\frac{\mathrm{Km}}{\mathrm{GeV}}\right)\right)}{1 - 1.52P\left(\left\langle \frac{L}{E_\nu} \right\rangle = 10^2 \left(\frac{\mathrm{Km}}{\mathrm{GeV}}\right)\right)} , \tag{26}$$

since the average values of $\langle L/E_\nu \rangle$ for the neutrinos which produce the upward stopping muons and the upward through-going muons are given by $\sim 10^3$ (Km/GeV) and $\sim 10^2$ (Km/GeV), respectively. If the oscillation is already in the rapid oscillation region (*i.e.*, $\Delta m^2_{32} \gtrsim 10^{-2}\mathrm{eV}^2$) the above ratio is, of course, unity. However, if that is not the case, the ratio can be different from unity. For example, the above factor takes the following values,

$$R\left(\frac{S}{T}\right) = \begin{cases} 0.38 & \text{for} \quad \Delta m^2_{32} = 10^{-3}\mathrm{eV}^2 \\ 0.82 & \text{for} \quad \Delta m^2_{32} = 3 \times 10^{-3}\mathrm{eV}^2 \end{cases} , \tag{27}$$

which can soon be tested in the future.

3 See-Saw Enhancement of Neutrino Mixing

So far we have conjectured that three neutrinos are maximally mixed, motivated by the vacuum oscillation solutions for the solar neutrino and atmospheric neutrino problems. The simultaneous solution requires the neutrino mass hierarchy , $m_1 \ll m_2 \ll m_3$ and the maximal mixings.

The obvious question then is "Is it possible for completely non-degenerate (in mass) neutrinos to have maximal, or at least very large mixings ?" The conventional wisdom tells us that mixing angles are determined by lepton mass ratios. In the quark sector, the Cabibbo angle is well reproduced by the formula $\tan^2\theta_C = m_d/m_s$. In the same spirit, the neutrino mixing angle, in the case of two generations, is expected to be determined by the ratios m_e/m_μ and $m(\nu_1)/m(\nu_2)$. If $m(\nu_1) \ll m(\nu_2)$, one expects the neutrino mixing angle to be very small.

However, this conclusion is not valid when one invokes the see-saw mechanism. The possibility that the see-saw mechanism may enhance lepton mixing up to maximal was first discussed by Smirnov[23]. When the see-saw mechanism is invoked with more than one generation of neutrinos, mixing angles for the resultant light Majorana neutrinos become dependent upon masses of heavy right-handed Majorana neutrinos which are usually provided by GUTS models. In particular, when certain relationships among the mass parameters in the original Dirac and heavy right-handed Majorana mass matrices are satisfied, mixing angles for the light Majorana neutrinos can substantially be enhanced. Let us consider the 4×4 mass matrix, (*i.e.*, the case of the two generations)

$$\mathcal{M} = \begin{pmatrix} \mathbf{0} & \mathbf{m} \\ \mathbf{m} & \mathbf{M} \end{pmatrix} , \tag{28}$$

where $\mathbf{m}$ and $\mathbf{M}$ are 2×2 real and symmetric matrices, representing the Dirac and heavy Majorana neutrino mass matrices, respectively.

Let us define $U(\theta_{_D})$ and $U(\theta_{_M})$ such that

$$\begin{aligned} U^T(\theta_{_D})\,\mathbf{m}\,U(\theta_{_D}) &= \hat{\mathbf{m}} , \\ U^T(\theta_{_M})\,\mathbf{M}\,U(\theta_{_M}) &= \hat{\mathbf{M}} , \end{aligned} \tag{29}$$

where $\hat{\mathbf{m}}$ and $\hat{\mathbf{M}}$ are 2×2 diagonal matrices. By block-diagonalizing the matrix $\mathcal{M}$, we find, assuming $\mid m_{ij} \mid \ll \det\mathbf{M}$ in the spirit of the see-saw mechanism, the Majorana mass matrix for light neutrinos which are completely decoupled from super heavy ones as

$$\mathbf{m}_{\rm Maj} = \mathbf{m}\ \mathbf{M}^{-1}\ \mathbf{m} \ . \tag{30}$$

Solving Eq. (29) for $\mathbf{m}$ and $\mathbf{M}$ and substituting them into Eq. (30), we obtain

$$\begin{aligned} \mathbf{m}_{\mathrm{Maj}} &= \left[U(\theta_{_D})\hat{\mathbf{m}}U^T(\theta_{_D})\right]\left[U(\theta_{_M})\hat{\mathbf{M}}^{-1}U^T(\theta_{_M})\right]\left[U(\theta_{_D})\hat{\mathbf{m}}U^T(\theta_{_D})\right] \\ &\equiv U(\theta_{_D})\mathbf{m}_{ss}U^T(\theta_{_D}) \ , \end{aligned} \tag{31}$$

where

$$\mathbf{m}_{ss} \equiv \hat{\mathbf{m}}U(\theta_{_M}-\theta_{_D})\hat{\mathbf{M}}^{-1}U^T(\theta_{_M}-\theta_{_D})\hat{\mathbf{m}} \ , \tag{32}$$

and we have used $U^T(\alpha)U(\beta) = U^T(\alpha-\beta)$ and $U^T(\beta-\alpha) = U(\alpha-\beta)$. Defining the orthogonal matrix that diagonalizes $\mathbf{m}_{ss}$ in Eq. (32) as $U(\theta_{ss})$, we can rewrite Eq. (31) as

$$\mathbf{m}_{\mathrm{Maj}} = U(\theta_{_D}+\theta_{ss})\hat{\mathbf{m}}_{ss}U^T(\theta_{_D}+\theta_{ss}) \ , \tag{33}$$

implying that the overall mixing angle θ_ν is now

$$\theta_\nu = \theta_{_D} + \theta_{ss} \ . \tag{34}$$

Therefore, even if the angle $\theta_{_D}$ is originally small, the neutrino mixing angle can be large if the angle due to the see-saw mechanism is large.

We now explicitly demonstrate the possibility of a large $\theta_\nu = \theta_{_D} + \theta_{ss}$. In order to illustrate the case in point in a more transparent way than originally presented by Smirnov, let us consider following mass matrix without imposing any restrictions on $\mathbf{m}$ and $\mathbf{M}$.

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & m_1 & m \\ 0 & 0 & m & m_2 \\ m_1 & m & M_1 & M \\ m & m_2 & M & M_2 \end{pmatrix} \ . \tag{35}$$

We are assuming, in the spirit of the see-saw mechanism,

$$\mid m \mid, \mid m_i \mid \ll \det\mathbf{M}. \tag{36}$$

After block-diagonalizing the matrix $\mathcal{M}$, we obtain

$$\begin{aligned}\mathbf{m}_{\rm Maj} = & \frac{1}{M_1M_2 - M^2} \\ & \times \begin{pmatrix} m_1^2M_2 - 2m_1mM + m^2M_1 & \begin{array}{r} m(m_1M_2 + m_2M_1) \\ - M(m_1m_2 + m^2) \end{array} \\ \begin{array}{r} m(m_1M_2 + m_2M_1) \\ - M(m_1m_2 + m^2) \end{array} & m_2^2M_1 - 2m_2mM + m^2M_2 \end{pmatrix} .\end{aligned} \tag{37}$$

It is straight-forward to calculate $\hat{\mathbf{m}}_{ss} \equiv \mathrm{diag}(\lambda_1, \lambda_2)$ and $U(\theta_{_D} + \theta_{ss})$. The results are

$$\begin{aligned}\lambda_{1,2} \;=\; & \frac{1}{2(M_1M_2 - M^2)} \left\{ \begin{array}{l} m_1^2M_2 - 2mm_1M + m^2M_1 \\ \qquad + m_2^2M_1 - 2mm_2M + m^2M_2 \end{array} \right. \\ & \left. \pm \left[\begin{array}{l} [(m_1^2M_2 - 2mm_1M + m^2M_1) - (m_2^2M_1 - 2mm_2M + m^2M_2)]^2 \\ \qquad + 4[m(m_1M_2 + m_2M_1) - M(m_1m_2 + m^2)]^2 \end{array} \right]^{\frac{1}{2}} \right\} ,\end{aligned} \tag{38}$$

and

$$\tan\left[2(\theta_{_D} + \theta_{ss})\right] = \frac{2\left[m(m_1M_2 + m_2M_1) - M(m_1m_2 + m^2)\right]}{(m_2^2M_1 - 2mm_2M + m^2M_2) - (m_1^2M_2 - 2mm_1M + m^2M_1)} . \tag{39}$$

The most obvious condition for the maximal mixing and mass hierarchy is that the four elements of the matrix $\mathbf{m}_{\rm Maj}$ be the same. In other words, the matrix elements of $\mathbf{m}$ and $\mathbf{M}$ should satisfy

$$(m_2^2 - m^2)\left(\frac{M_1}{M_2}\right) + 2m(m_1 - m_2)\left(\frac{M}{M_2}\right) = m_1^2 - m^2 , \tag{40}$$

and

$$(m_2 - m)^2\left(\frac{M_1}{M_2}\right) + 2(m_1 - m)(m_2 - m)\left(\frac{M}{M_2}\right) = -(m_1 - m)^2 . \tag{41}$$

where M_2 is assumed to be finite. Eq. (40) comes from matching two diagonal elements of the mass matrix in Eq. (37) and Eq. (41) is obtained by equating the sum of diagonal elements and that of off-diagonal elements. Now we can solve the equations for M_1/M_2 and M/M_2. We find, from Eqs. (40) and (41),

$$\frac{M_1}{M_2} = \frac{(m-m_1)^2}{(m-m_2)^2} \quad , \quad \frac{M}{M_2} = -\frac{m-m_1}{m-m_2} \, , \tag{42}$$

for which, the mixing angle $(\theta_{_D} + \theta_{ss})$ becomes 45^0, and $\lambda_1 = 0$ and λ_2 is finite, as desired. It is clear, however, that in this case the determinant of $\mathbf{M}$ vanishes so that the inverse of the matrix does not exist. To remedy this problem, we have to relax one or both conditions of Eqs. (40) and (41). Since the condition of Eq. (40) is directly related to the maximal mixing, one is forced to relax Eq. (41).

Now, with one less condition to satisfy, we will give several examples in which the maximal mixing is realized with significant hierarchy in the mass eigenvalues.

Case 1. The first case is for $M = 0$. The obvious possibility in this case is that all the elements of $\mathbf{m}$ are identical. The mass eigenvalues of light Majorana neutrinos are $(0, 2m^2(M_1+M_2)/M_1M_2)$ and they have extreme hierarchy, *i.e.*, one of the eigenvalues is zero and the other finite. In this case, the see-saw mechanism provides only the smallness of the light Majorana neutrinos while the maximal mixing with the mass hierarchy is the result of the nature of the matrix $\mathbf{m}$.

Case 2. The next possibility is when M_1 and M_2 are adjusted with still keeping $M = 0$ so that the condition $\tan\left[2(\theta_{_D} + \theta_{ss})\right] = \infty$ is met, *i.e.*,

$$\frac{M_1}{M_2} = \frac{m^2 - m_1^2}{m^2 - m_2^2} \, .$$

Then, the ratio of two eigenvalues is given by

$$\frac{\lambda_1}{\lambda_2} = \frac{(m_1 - m)(m_2 - m)}{(m_1 + m)(m_2 + m)} \, . \tag{43}$$

When $m_1 \simeq m$ and/or $m_2 \simeq m$, λ_1/λ_2 can be very small and M_1 and M_2 can have a very wide range of ratios.

Case 3. Another case of interest is when the neutrino Dirac mass matrix has the Fritzsch form, *i.e.,* $m_1 = 0$. Equation (40) then reduces to

$$(m_2^2 - m^2)M_1 - 2mm_2M + m^2M_2 = 0 \ , \tag{44}$$

and the hierarchical condition reduces to

$$mM_1 \simeq \mid m_2M_1 - mM \mid \tag{45}$$

Equation (44) guarantees the maximal mixing and Eq. (45) implies a desired mass hierarchy.

4 Conclusion

Motivated by recent indications of the possible large mixing angle solutions for both the Solar and atmospheric neutrino problems, we have examined the scheme in which all three neutrinos are maximally mixed. In this model, only the neutrino masses are treated as unknown parameters. We have analyzed the most recent data on the Solar and atmospheric neutrinos using this model. All three types of the Solar neutrino data (Cl, Water, Ga) can be explained by vacuum oscillations of the maximally mixed three generations of neutrinos if $\Delta m_{21}^2 \equiv m_2^2 - m_1^2 \simeq 10^{-10}\mathrm{eV}^2$ and $\Delta m_{32}^2 \gg 10^{-10}\mathrm{eV}^2$, as suggested by analysis based on the two generation treatment. That is, addition of the third generation of neutrino does not change the prediction of the two generation analysis. One definite prediction of the three generation model that the observed event rate for the Ga experiment cannot be larger than 80 SNU can soon be tested as the Ga data improves in the near future.

Similarly, the current data on the atmospheric neutrino experiments can be explained in this model for $10^{-3}\mathrm{eV}^2 \lesssim \Delta m_{32}^2 \lesssim (10^{-1} \sim 1)\mathrm{eV}^2$ and $\Delta m_{21}^2 \simeq 10^{-10}\mathrm{eV}^2$. It is interesting to note that in spite of the equal values of $P(\nu_e \to \nu_\mu)$, $P(\nu_\mu \to \nu_\tau)$ and $P(\nu_e \to \nu_\tau)$ in our model, the atmospheric $\nu_e(\overline{\nu_e})$ flux naturally remains the same whereas that of $\nu_\mu(\overline{\nu_\mu})$ is substantially suppressed as indicated by the data.

We have demonstrated that although the conventional wisdom learned from the quark sector suggests small mixing angles in the lepton sector, it is possible to have large or maximal mixings for the neutrinos if the see-saw mechanism is invoked to generate small neutrino masses. We have presented explicit two-generation examples (or models) in which the two light Majorana neutrinos are maximally mixed even though their masses are highly non-degenerate.

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Figure Caption

Fig. 1: Comparison of Calculated and Observed Event rates for Kamioka, Cl and Ga Experiments.

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Figure 1

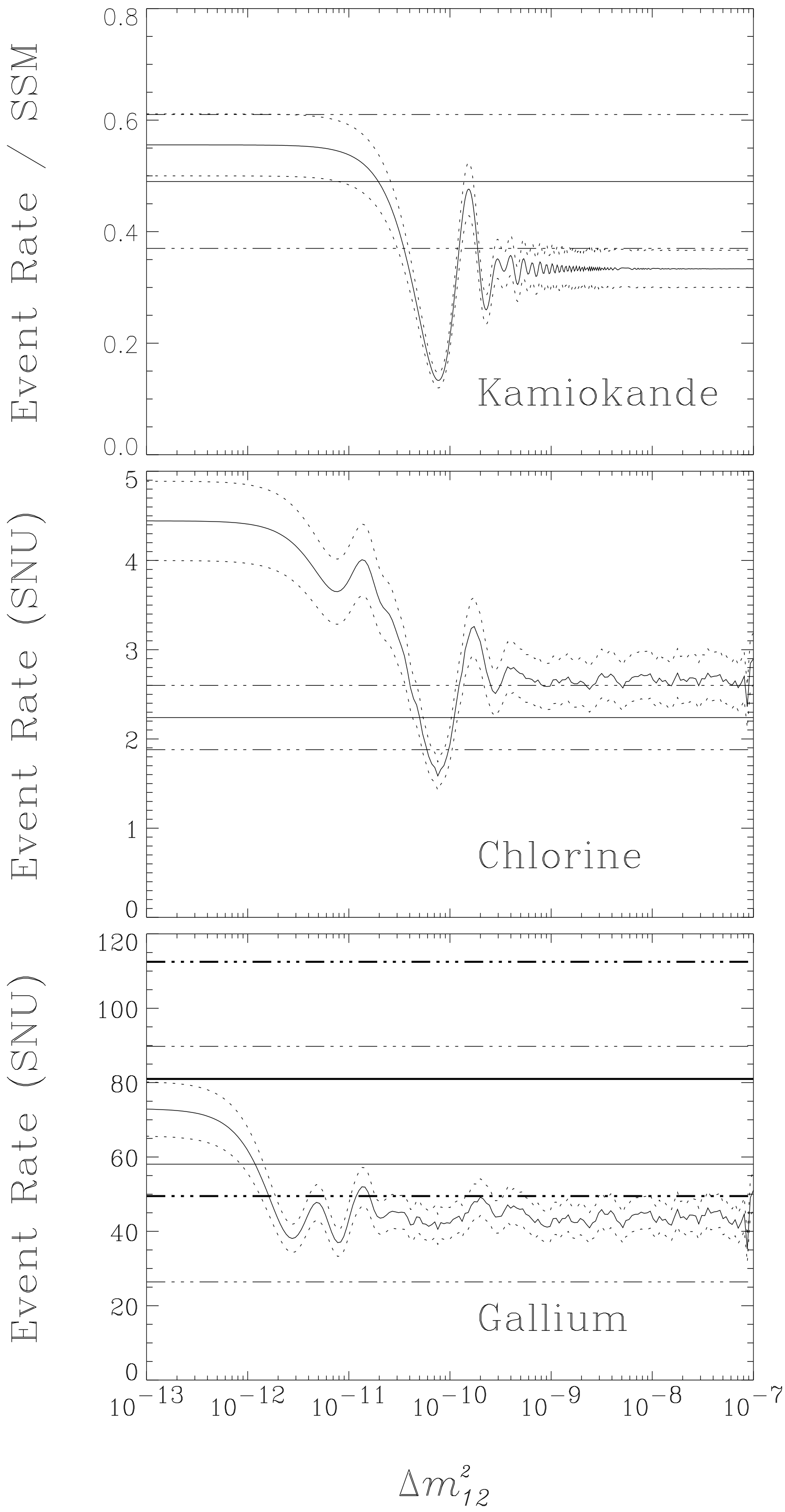
Event Rate / SSM
Kamiokande
Event Rate (SNU)
Chlorine
Event Rate (SNU)
Gallium
Δm^2_{12}

Figure 1